Math 249 Lecture 12 Notes

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1 Inner Products of Symmetric Functions

1.1 Inner products involving h_λ, m_λ, and e_λ

Last time, we proved a Cauchy identity for p_λ:

$$\sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda} = \Omega[XY] = \prod_{i,j} \frac{1}{1-x_iy_j}$$

Let's prove a Cauchy identity for h_λ and m_λ.

Proposition 1.1.

$$\Omega[XY] = \sum_\lambda h_\lambda(x)m_\lambda(y)$$

Proof. The coefficient of $m_\lambda(y)$ in the expression $\prod_{i,j}(1-x_iy_j)^{-1}$ is the coefficient of $y^{\lambda_1}y^{\lambda_2}\cdots y^{\lambda_\ell}$ in the expression $\prod_j \left(\prod_i (1-x_iy_j)^{-1}\right)$. This inner term is $H(y_j) = \sum h_n(x)y_j^n$, so we get the product $h_{\lambda_1(x)}h_{\lambda_2(x)}\cdots h_{\lambda_\ell(x)} = h_\lambda(x)$. □

This shows that $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$.

Proposition 1.2. *The map $\omega : \Lambda \to \Lambda$ is an isometry.*

Proof. Apply ω to $\langle p_\lambda, p_\mu\rangle$.

$$\langle \omega(p_\lambda), \omega(p_\mu)\rangle = \varepsilon(\sigma_\lambda)\varepsilon(\sigma_\mu)\langle p_\lambda, p_\mu\rangle = \varepsilon(\sigma_\lambda)\varepsilon(\sigma_\mu)\delta_{\lambda,\mu} = \delta_{\lambda,\mu} = \langle p_\lambda, p_\mu\rangle .$$

Since the power sum symmetric functions form a basis for Λ, we are done. □

Applying ω to $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ gives us that $\langle e_\lambda, \omega(m_\mu)\rangle = \delta_{\lambda,\mu}$. Denoting the coefficient of the term m_μ as $\langle m_\mu\rangle$, we also have $\langle h_\lambda, h_\mu\rangle = \langle m_\mu\rangle h_\lambda = b_{\lambda,\mu} = b_{\mu,\lambda}$.

1.2 Inner products of h_λ and p_μ

Let V be a H-module, where $H \subseteq G$. We can construct $\mathbb{C}G \otimes_{\mathbb{C}H} V$, which is a $\mathbb{C}G$ module, where $x \otimes ay = xa \otimes y$ if $a \in H$.

Definition 1.1. Let V be a H-module, where $H \subseteq G$. The *induced character* $\chi \uparrow_H^G$ is the character of $\mathbb{C}G \otimes_{\mathbb{C}H} V$.

Evaluating this character in general requires a bit of work involving coset representatives g_i of cosets $g_h \in G/H$ that permute basis elements v_i of V. Instead, we will focus on a very simple case. Let V be the trivial representation, so we get $\mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}$. Tensoring over $\mathbb{C}H$ makes $gh \otimes 1 = g \otimes h(1) = g \otimes 1$. Then this is isomorphic to $\mathbb{C}(G/H)$.

Let $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$, thought of as a subset of S_n, and define the character $\mathbb{1}_{S_\lambda} \uparrow^{S_n}$, where $\mathbb{1}$ is the trivial representation. Then $\mathbb{1}_{S_\lambda} \uparrow^{S_n} = \chi_{\mathbb{C}(S_n/S_\lambda)}$. To evaluate this character, note that the action of S_n on $\mathbb{C}(S_n/S_\lambda)$ is the same as the action of S_n permuting the letters of words $a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_\ell^{\lambda_\ell}$ (λ_1 copies of the letter a_1, etc.). Then

$$\mathbb{1}_{S_\lambda} \uparrow^{S_n} (\sigma) = \text{ number of words in } a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_\ell^{\lambda_\ell} \text{ fixed by } \sigma.$$

Proposition 1.3. *Let F be the Frobenius characteristic map. Then*

$$F(\mathbb{1}_{S_\lambda} \uparrow^{S_n}) = h_\lambda.$$

Proof. We want to prove that $\langle F(\mathbb{1}_{S_\lambda} \uparrow^{S_n}), p_\mu \rangle = \langle h_\lambda, p_\mu \rangle$ for each λ, μ. The former is the same as $\langle \mathbb{1}_{S_\lambda} \uparrow^{S_n}, \delta_\mu \rangle = \mathbb{1}_{S_\lambda} \uparrow^{S_n} (\sigma_\mu)$ for σ_μ a permutation of S_n with cycle structure $\mu = (\mu_1, \ldots, \mu_k)$. So as we argued before, this is the number of number of words in letters $a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_\ell^{\lambda_\ell}$ fixed by σ_μ. This amounts to mapping the indices μ_i to λ_j (via some $f : [k] \to [\ell]$) such that $\sum_{i \in f^{-1}(\{j\})} \mu_i = \lambda_j$; i.e. we count the number of refinements of the partition λ to the partition μ.

$$h_n = \sum_{|\lambda|=n} \frac{p_\lambda}{z_\lambda} = F(\mathbb{1}_{S_\lambda} \uparrow^{S_n})$$

What about $\langle h_\lambda, p_\mu \rangle$? Since $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$, $\langle h_\lambda, p_\mu \rangle = \langle x^\lambda \rangle p_\mu$, the coefficient of m_λ in p_μ. Finding this coefficient is the same process as earlier; if we are trying to map the x_j back to p_{μ_i} where they came from, we are finding the number of mappings of the indices μ_i to λ_j (via some $f : [k] \to [\ell]$) such that $\sum_{i \in f^{-1}(\{j\})} \mu_i = \lambda_j$. This completes the proof. $\square$

2 Antisymmetric functions

2.1 Antisymmetric functions and related partitions

We want to introduce the Schur functions, which will be another basis for the symmetric functions. Instead of starting with a combinatorial definition, we'll present a classical definition in n variables first. We need some background.

Definition 2.1. A function $f(x_1, \ldots, x_n)$ is *antisymmetric* if for any permutation $\sigma \in S_n$, $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \varepsilon(\sigma) f(x_1, \ldots, x_n)$.

If a monomial has two variables with the same power, then the transposition switching those two variables will make the coefficient of the monomial equal to the negative of itself; this makes the coefficient 0. So if you have a monomial with nontrivial stabilizer in S_n, then it must have coefficient 0. This means that all antisymmetric functions must have terms with exponents that correspond to partitions λ with no repeated λ_i.

Let $\rho = (n-1, n-2, \ldots, 1, 0)$, and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. Then $\lambda + \rho = (\lambda_1 + n - 1 > \lambda_2 + n - 2 > \cdots > \lambda_n)$ is a strictly decreasing sequence. Conversely, given any strictly decreasing sequence, we can subtract ρ and get a weakly decreasing sequence.

2.2 Monomial antisymmetric functions and the Vandermonde determinant

Definition 2.2. The *monomial antisymmetric functions* are $a_{\lambda+\rho} = x^{\lambda+\rho} \pm$ similar terms obtained by permuting the variables.

These are a basis for the antisymmetric functions, $\mathbb{Z}[x_1, \cdots, x_n]^{\varepsilon}$. We can also express them as

$$\begin{aligned} a_{\lambda+\rho} &= \sum_{\sigma \in S_n} \varepsilon(\sigma) \sigma(x^{\lambda+\rho}) \\ &= \det \begin{bmatrix} x_1^{(\lambda+\rho)_1} & x_1^{(\lambda+\rho)_2} & \cdots & x_1^{(\lambda+\rho)_n} \\ x_2^{(\lambda+\rho)_1} & x_2^{(\lambda+\rho)_2} & \cdots & x_2^{(\lambda+\rho)_n} \\ \vdots & \vdots & & \vdots \\ x_n^{(\lambda+\rho)_1} & x_n^{(\lambda+\rho)_2} & \cdots & x_2^{(\lambda+\rho)_n} \end{bmatrix}. \end{aligned}$$

Similarly, we can make the following definition.

Definition 2.3. The *Vandermonde determinant is*

$$a_\rho = \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{bmatrix} = \prod_{i<j} (x_i - x_j).$$

Why does the 2nd equality hold? The left hand side is a multiple of the right hand side because the left hand side is a polynomial that equals 0 whenever $x_i = x_j$ for $i \neq j$. The degrees are equal, so the left hand side must be a constant multiple c of the right; checking the coefficient in front of a term shows that $c = 1$, so the two expression are equal.

In fact, any antisymmetric function is a multiple of the Vandermonde determinant because it must be 0 whenever $x_i = x_j$ for $i \neq j$.